Student Number ______________________

Ascham School

VI·ET·ANIMO

Mathematics Extension 1

Trial HSC Examination

Monday 25th July 2016

2 hours

General Instructions

- Reading time – 5 minutes.
- Working time – 2 hours.
- Write using black pen.
- Board-approved calculators may be used.
- A Reference Sheet is provided.
- In Questions 11–14, show relevant mathematical reasoning and/or calculations.

Total marks – 70

Section I Pages 2–4

10 marks

- Use the Multiple Choice Answer Sheet provided to answer Q1-10.
- Allow about 15 minutes for this section.

Section II Pages 5–9

60 marks

- Answer Questions 11-14.
- Answer each question in a new booklet.
- Label all sections clearly with your name/number and teacher's initials.
- Allow about 1 hour and 45 minutes for this section.

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

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1 If α, β and γ are roots of the equation $x^3 - 3x^2 + 4x + 2 = 0$, find the value of $\dfrac{1}{\alpha\beta} + \dfrac{1}{\beta\gamma} + \dfrac{1}{\alpha\gamma}$.

(A) 4 (B) $\dfrac{1}{4}$ (C) $-\dfrac{2}{3}$ (D) $-\dfrac{3}{2}$

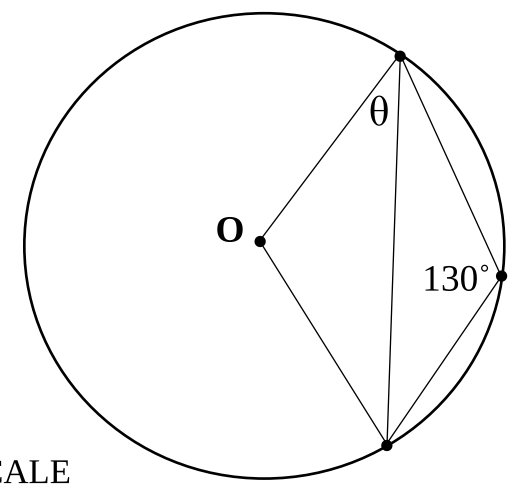
2 O is the centre of the circle below. Which is the correct value for θ?

(A) 25°

(B) 40°

(C) 50°

(D) 65°



NOT TO SCALE

3 What are the asymptotes of $y = \dfrac{1-x^2}{x^2-4}$?

(A) $x = 0, \quad x = 2, \quad y = -1$ (B) $x = 2, \quad x = -2, \quad y = 1$

(C) $x = 2, \quad x = -2, \quad y = -1$ (D) $x = 2, \quad x = -2, \quad y = 0$

4 The equation of the normal to the curve $x^2 = 20y$ at the point $(10p,\ 5p^2)$ is:

(A) $x + py = 5p^3 + 10p$

(B) $x - py = 5p^3 - 10p$

(C) $px + y = 15p^2$

(D) $px - y + 15p^2 = 0$

5 Choose the correct value of $\lim_{x \to 0} \dfrac{\sin \frac{x}{2}}{3x}$.

(A) $\dfrac{1}{6}$

(B) $\dfrac{1}{3}$

(C) $\dfrac{1}{2}$

(D) 6

6 The angle θ satisfies $\sin\theta = \dfrac{1}{\sqrt{5}}$ and $\dfrac{\pi}{2} < \theta < \pi$.

What is the exact value of $\sin 2\theta$?

(A) $\dfrac{4}{5}$

(B) $\dfrac{4}{\sqrt{5}}$

(C) $-\dfrac{4}{5}$

(D) $-\dfrac{4}{\sqrt{5}}$

7 A particle has displacement function $x = 3\cos 5t$. Its acceleration can be written as:

(A) $\ddot{x} = 9x$

(B) $\ddot{x} = -9x$

(C) $\ddot{x} = 25x$

(D) $\ddot{x} = -25x$

8 The velocity, v metres per second, of a particle moving in simple harmonic motion along the x-axis is given by the equation $v^2 = 100 - 16x^2$.

What is the amplitude, in metres, of the motion of the particle?

(A) $\dfrac{2}{5}$

(B) $2\dfrac{1}{2}$

(C) 4

(D) 10

9 Select the equation which could represent the graph below:

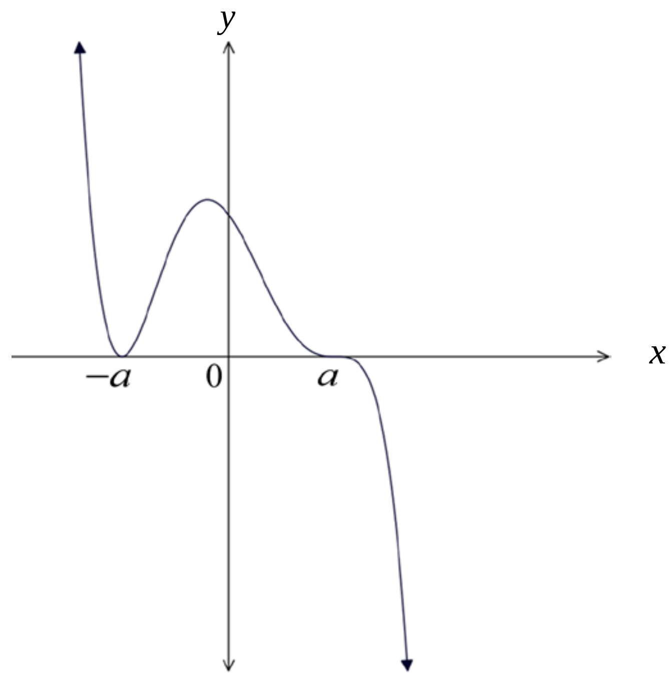
DIAGRAM NOT TO SCALE

(A) $y=(x+a)^2(a-x)^3$

(B) $y=(x-a)^2(x+a)^3$

(C) $y=(x+a)^2(x-a)^3$

(D) $y=(x-a)^2(a-x)^3$



10 Which of the following represents the derivative of $\cos^{-1}\left(\frac{1}{x}\right)$?

(A) $\dfrac{1}{\sqrt{x^2-1}}$

(B) $\dfrac{-1}{\sqrt{x^2-1}}$

(C) $\dfrac{1}{x\sqrt{x^2-1}}$

(D) $\dfrac{-1}{x\sqrt{x^2-1}}$

Section II

60 marks
Attempt Questions 11–14
Allow about 1 hour and 45 minutes for this section

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.
In Questions 11–14, you should include relevant mathematical reasoning and/ or calculations.

Question 11 (15 marks) Use a SEPARATE writing booklet.

(a) Find the value of k if $x^3 + 3x^2 - 4x + k$ is divisible by $x + 2$. **2**

(b) Solve the inequality $\dfrac{x}{x-5} \geq 2$. **3**

(c) The lines $y = mx + 5$ and $y = -3x + 7$ are inclined to each other at an angle of 45°.

(i) Show that $\left|\dfrac{m+3}{1-3m}\right| = 1$. **1**

(ii) Hence find the possible values of m. **2**

(d) Find the general solutions to the equation: $\sin 2\theta = \sin\theta$. **3**

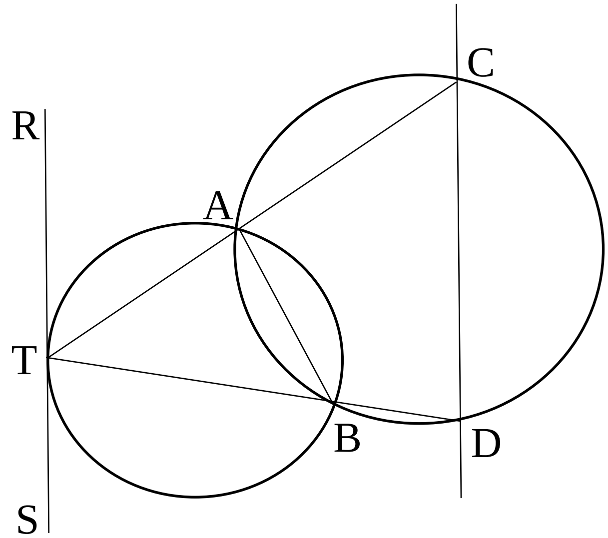
(e) Consider the graph of the function $f(x) = x^2 + 2x$ for $x \leq -1$.

(i) Find the equation of $y = f^{-1}(x)$ and state its domain. **2**

(ii) Sketch the graphs of $y = f(x)$ and its inverse function $y = f^{-1}(x)$ on the same number plane. **2**

Question 12 (15 marks) Use a SEPARATE writing booklet.

(a) In the diagram, two unequal circles intersect at A and B. The line RS is tangential to the smaller circle at T. The lines TA and TB meet the larger circle at C and D respectively.



NOT TO SCALE

Copy the diagram into your exam booklet.

(i) State a theorem to explain why $\angle BAT = \angle BDC$. **1**

(ii) Prove that $RS \parallel CD$. **3**

(b) Consider the equation $x^2 - 9 + \log_e x = 0$.

(i) By drawing the graph of $y = \log_e x$ and another appropriate graph on the same axes, explain why the equation has only one root. **1**

(ii) Show, using calculations, that the root of the equation lies between 2 and 3. **2**

(iii) Taking $x_0 = 2.5$ as the first approximation, use Newton's method to find a second approximation of the root correct to 2 decimal places. **2**

(c) (i) Use the substitution $x = \sin\theta$ to show that $\displaystyle\int_0^{\frac{1}{2}} \frac{x^2}{\sqrt{1-x^2}}\,dx = \int_0^{\frac{\pi}{6}} \sin^2\theta\ d\theta$. **3**

(ii) Hence find the exact value of $\displaystyle\int_0^{\frac{1}{2}} \frac{x^2}{\sqrt{1-x^2}}\,dx$. **3**

Question 13 (15 marks) Use a SEPARATE writing booklet.

(a) (i) Show that $\sqrt{3}\cos\theta + \sin\theta = 2\cos\left(\theta - \frac{\pi}{6}\right)$. **2**

(ii) Hence solve the equation $\sqrt{3}\cos\theta + \sin\theta = -1$ for $0 \le \theta \le 2\pi$. **3**

A particle moves in simple harmonic motion along a straight line so that its displacement, x metres, at time t seconds, is given by:

$$x = \sqrt{3}\cos\left(\frac{t}{3}\right) + \sin\left(\frac{t}{3}\right).$$

(iii) Find the smallest positive value of t for which $x = -1$. **1**

(iv) Find the distance travelled by the particle in going from its initial position to the position $x = -1$. Justify your answer. **3**

(b) The velocity of a particle is given by the equation $v = \frac{1}{e^x}$.

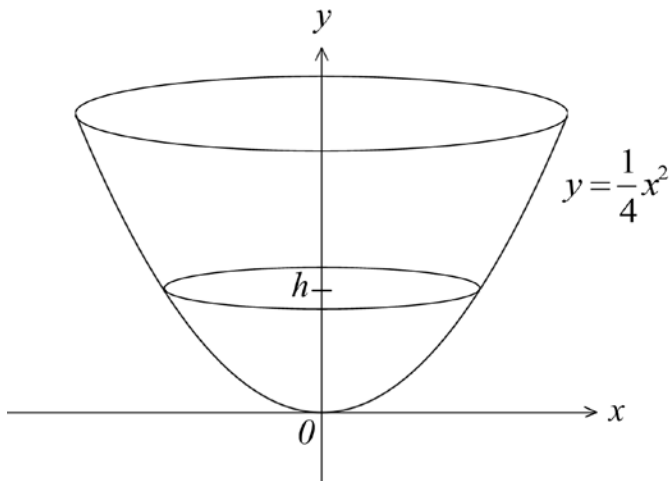
If the initial displacement is $x = 0$, find the equation for the displacement x, in terms of t. **3**

(c) Prove by mathematical induction that $\sum_{r=1}^{n} \log\left(\frac{r+1}{r}\right) = \log(n+1)$. **3**

Question 14 (15 marks) Use a SEPARATE writing booklet.

(a) A large industrial container is in the shape of a paraboloid, which is formed by rotating part of the parabola $y = \frac{1}{4}x^2$ around the y-axis, as shown in the diagram.

Liquid is poured into the container at the rate of $2\,\text{m}^3$ per minute.



(i) Show that the volume V m^3 of liquid in the container when the depth of liquid is h metres, is given by $V = 2\pi h^2$. **1**

(ii) At what rate is the height (h) of the liquid rising when the depth is 0.5 metres? **3**

(iii) If the container is 3 metres high, how long will it take to fill the container? **1**

(b) $P(2t,\ t^2)$ is a point on the parabola $x^2 = 4y$ with focus S (0, 1).

The point M divides the interval SP externally in the ratio 3 **:** 1.

(i) Show that the coordinates of the point M are $\left(3t,\ \dfrac{3t^2-1}{2}\right)$. **2**

(ii) Hence show that as P moves on the parabola $x^2 = 4y$, then M moves on the parabola $x^2 = 6y + 3$. **2**

Question 14 continues on page 9

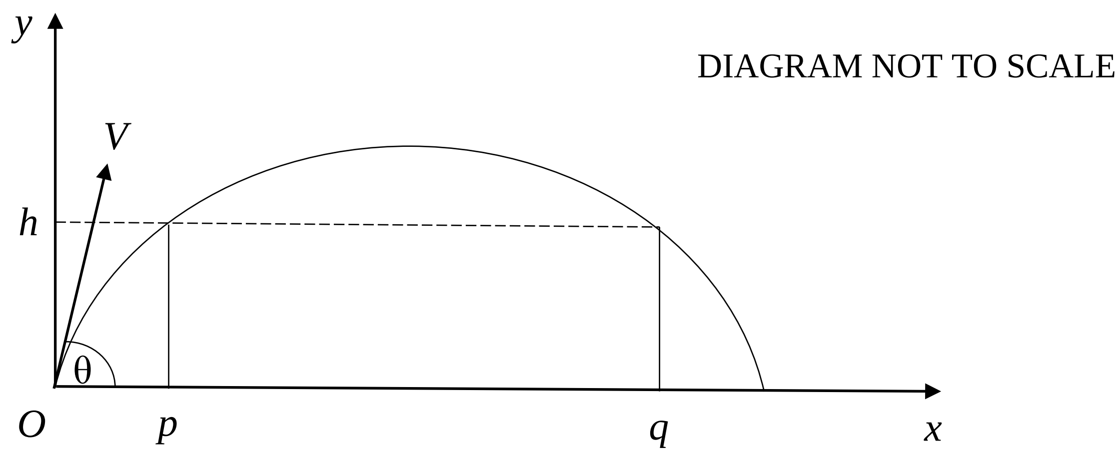
(c) A particle is projected with initial velocity V ms^{-1} at an angle of elevation θ from the origin O. The particle just clears two vertical chimneys of height h metres at horizontal distances p metres and q metres from O.

The diagram shows the path of this projectile where the acceleration due to gravity is taken as $10\,\text{ms}^{-2}$ and air resistance is ignored.

You may assume that the projectile's trajectory is defined by the equations:

$$x = Vt\cos\theta \qquad \text{and} \qquad y = Vt\sin\theta - 5t^2 \qquad \text{[Do not prove these.]}$$

where x and y are the horizontal and vertical displacements of the projectile in metres at time t seconds.



(i) Show that $V^2 = \dfrac{5p^2\left(1+\tan^2\theta\right)}{p\tan\theta - h}$. **3**

(ii) Hence show that $\tan\theta = \dfrac{h\left(p+q\right)}{pq}$. **3**

END OF PAPER

SOLUTIONS 2016 Extension 1 Maths Trial Exam

SECTION I

(1) $\frac{1}{\alpha\beta} + \frac{1}{\beta\gamma} + \frac{1}{\alpha\gamma}$

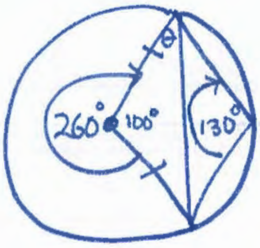
$= \frac{\gamma + \alpha + \beta}{\alpha\beta\gamma}$

$= \frac{-b/a}{-d/a}$ where $a = 1$, $b = -3$, $c = 4$, $d = 2$

$= \frac{--3/1}{-2/1}$

$= \frac{-3}{2}$ $\therefore$ D

(2)



$\theta = \frac{180° - 100°}{2}$

$\theta = 40°$

$\therefore$ B

(3) $y = \frac{1 - x^2}{(x-2)(x+2)}$

$x - 2 \neq 0$

$x + 2 \neq 0$

$\therefore$ Vertical Asymptotes at $x = \pm 2$

$\lim_{x \to \infty} \frac{1 - x^2}{x^2 - 4} = \lim_{x \to \infty} \frac{1/x^2 - x^2/x^2}{x^2/x^2 - 4/x^2}$

$= \lim_{x \to \infty} \frac{1/x^2 - 1}{1 - 4/x^2}$

$= \frac{0 - 1}{1 - 0}$

$= -1$

$\therefore$ Horizontal Asymptote at $y = -1$

$\therefore$ C

(4) $x^2 = 20y$ or $y = \frac{x^2}{20}$

$\frac{dy}{dx} = \frac{x}{10}$

At $x = 10p$: $\frac{dy}{dx} = \frac{10p}{10} = p$

$\therefore$ Gradient of normal $= -\frac{1}{p}$

Equation of normal is:

$y - 5p^2 = \frac{-1}{p}(x - 10p)$

$py - 5p^3 = -x + 10p$

$\therefore x + py = 5p^3 + 10p$

$\therefore$ A

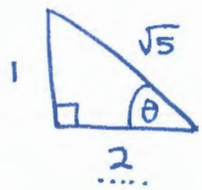
(5) $\lim_{x \to 0} \frac{\sin x/2}{3x}$

$= \lim_{x \to 0} \frac{\sin x/2}{x/2} \times \frac{x/2}{3x}$

$= 1 \times \frac{1}{6}$

$= \frac{1}{6}$ $\therefore$ A

(6) If $\sin\theta = \frac{1}{\sqrt{5}}$ and $\pi/2 < \theta < \pi$



$\cos\theta = -\frac{2}{\sqrt{5}}$

so $\sin 2\theta = 2\sin\theta\cos\theta$

$= 2 \times \frac{1}{\sqrt{5}} \times \frac{-2}{\sqrt{5}}$

$= \frac{-4}{5}$ $\therefore$ C

(7) $x = 3\cos 5t$

$\dot{x} = -15\sin 5t$

$\ddot{x} = -75\cos 5t$

$\ddot{x} = -25(3\cos 5t)$

$\ddot{x} = -25x$ $\therefore$ D

(8) when $v=0$:

$100-16x^2=0$

$(10-4x)(10+4x)=0$

$\therefore x=\pm\frac{5}{2}$ are endpoints

$\therefore$ centre of motion is $x=0$

and amplitude $=\frac{5}{2}$ $\therefore$ B

(9) $y=-(x+a)^2(x-a)^3$

$=(x+a)^2(a-x)^3$ $\therefore$ A

(10) $\frac{d}{dx}\left[\cos^{-1}(x^{-1})\right]$

$=\frac{-1}{\sqrt{1-(x^{-1})^2}}\times -x^{-2}$

$=\frac{-1}{\sqrt{1-\frac{1}{x^2}}}\times\frac{-1}{x^2}$

$=\frac{1}{x^2\sqrt{\frac{x^2-1}{x^2}}}$

$=\frac{1}{\frac{x^2}{\sqrt{x^2}}\sqrt{x^2-1}}$

$=\frac{1}{x\sqrt{x^2-1}}$ $\therefore$ C

SECTION II

Question 11

a) If $P(x)=x^3+3x^2-4x+k$

then $P(-2)=0$

$(-2)^3+3(-2)^2-4(-2)+k=0$

$-8+12+8+k=0$

$\therefore k=-12$

b) $\frac{x}{x-5}\geqslant 2$ $[x\neq 5]$

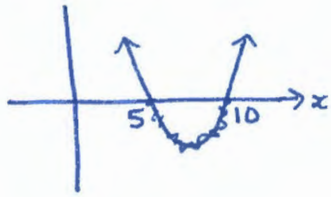
$\frac{x(x-5)^2}{x-5}\geqslant 2(x-5)^2$

$x(x-5)\geqslant 2(x-5)^2$

$2(x-5)^2-x(x-5)\leqslant 0$

$(x-5)[2(x-5)-x]\leqslant 0$

$(x-5)(x-10)\leqslant 0$



$\therefore 5<x\leqslant 10$

c) i) $\tan\theta=\left|\frac{m_1-m_2}{1+m_1m_2}\right|$

$\tan 45^\circ=\left|\frac{m--3}{1+m(-3)}\right|$

$1=\left|\frac{m+3}{1-3m}\right|$

$\therefore \left|\frac{m+3}{1-3m}\right|=1$

(as required)

(ii) $\dfrac{m+3}{1-3m} = 1$ or $\dfrac{m+3}{1-3m} = -1$

$m+3 = 1-3m$

$4m = -2$

$m = -\frac{1}{2}$

$m+3 = -(1-3m)$

$m+3 = -1+3m$

$4 = 2m$

$m = 2$

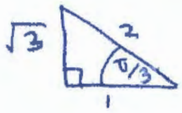
$\therefore$ $m = -\frac{1}{2}$ or $m = 2$

d) $\sin 2\theta = \sin\theta$

$2\sin\theta\cos\theta - \sin\theta = 0$

$\sin\theta(2\cos\theta - 1) = 0$

$\sin\theta = 0$ or $\cos\theta = \frac{1}{2}$



$\therefore$ $\theta = \pi n$ or $\pm\frac{\pi}{3} + 2n\pi$

$(n \in \mathbb{Z})$

e) i) Let $y = f(x)$

i.e. $y = x^2 + 2x$ where $x \leq -1$

$\therefore$ Inverse: $x = y^2 + 2y$

Completing the square...

$x + 1 = y^2 + 2y + 1$

$x + 1 = (y+1)^2$

$\pm\sqrt{x+1} = y + 1$

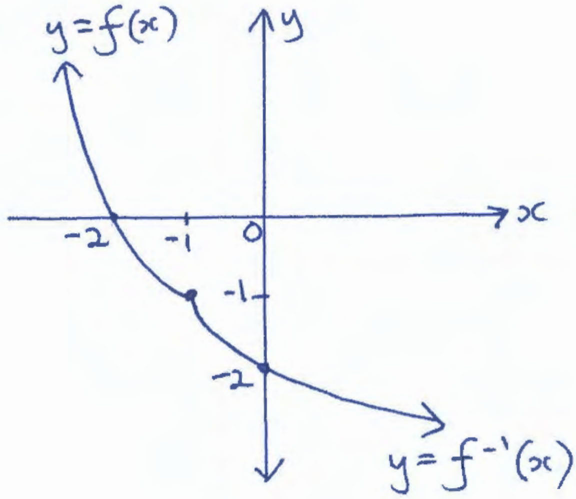
$y = -1 \pm \sqrt{x+1}$

but $y \leq -1$

$\therefore$ $f^{-1}(x) = -1 - \sqrt{x+1}$

over the domain $x \geq -1$

ii)



Question 12

a) i) The exterior ∠ of a cyclic quadrilateral equals the interior opposite ∠.

ii) $\angle BAT = \angle BTS$

(∠ between tangent and chord equals ∠ in alternate segment)

$\therefore \angle BTS = \angle BDC$

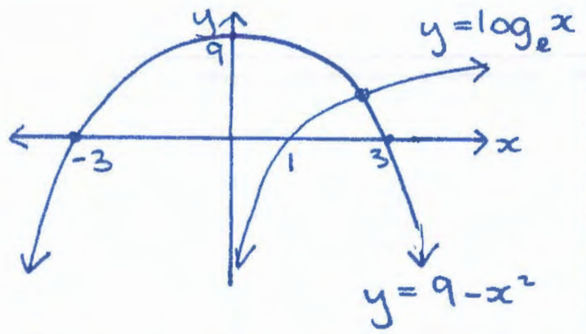
$\therefore RS \parallel CD$

(alternate ∠s equal)

b) i) $x^2 - 9 + \log_e x = 0$

or $\log_e x = 9 - x^2$

The solution represented by the x-value of the point of intersection of the graphs $y = \log_e x$ and $y = 9 - x^2$



As there is only one point of intersection, there is only one root of the equation $x^2 - 9 + \log_e x = 0$

ii) Let $f(x) = x^2 - 9 + \log_e x$

$f(2) = 2^2 - 9 + \log_e 2$

$= -4.306.... < 0$

$f(3) = 3^2 - 9 + \log_e 3$

$= 1.098.... > 0$

Since $f(2) < 0$ and $f(3) > 0$, the root must lie between 2 and 3, given $f(x)$ is continuous.

iii) If $f(x) = x^2 - 9 + \log_e x$

then $f'(x) = 2x + \frac{1}{x}$

Using: $\boxed{x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}}$

where $x_0 = 2.5$,

$x_1 = 2.5 - \frac{2.5^2 - 9 + \ln 2.5}{2(2.5) + \frac{1}{2.5}}$

$= 2.839....$

$\therefore \underline{x_1 \doteqdot 2.84}$ (2 decimal places)

c) i) If $x = \sin\theta$ then $\frac{dx}{d\theta} = \cos\theta$

i.e. $dx = \cos\theta \, d\theta$

and $\int \frac{x^2}{\sqrt{1-x^2}} dx = \int \frac{\sin^2\theta}{\sqrt{1-\sin^2\theta}} \cos\theta \, d\theta$

$= \int \frac{\sin^2\theta}{\sqrt{\cos^2\theta}} \cos\theta \, d\theta$

$= \int \sin^2\theta \, d\theta$

Also when $x = 0$: $\sin\theta = 0$

$\therefore \theta = 0$

and when $x = \frac{1}{2}$: $\sin\theta = \frac{1}{2}$

$\therefore \theta = \frac{\pi}{6}$

$\therefore \int_0^{\frac{1}{2}} \frac{x^2}{\sqrt{1-x^2}} dx = \int_0^{\pi/6} \sin^2\theta \, d\theta$

ii) $\int_0^{\frac{1}{2}} \frac{x^2}{\sqrt{1-x^2}} dx$

$= \int_0^{\pi/6} \sin^2\theta \, d\theta = \frac{1}{2}\int_0^{\pi/6} 1 - \cos 2\theta \, d\theta$

$= \frac{1}{2}\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\pi/6}$

$$= \frac{1}{2}\left[\frac{\pi}{6} - \frac{\sin(2\times \pi/6)}{2} - \left(0 - \frac{\sin(2\times 0)}{2}\right)\right]$$

$$= \frac{1}{2}\left[\frac{\pi}{6} - \frac{\sqrt{3}/2}{2} - 0\right]$$

$$= \frac{1}{2}\left[\frac{\pi}{6} - \frac{\sqrt{3}}{4}\right]$$

$$= \frac{1}{2} \times \frac{1}{2}\left[\frac{\pi}{3} - \frac{\sqrt{3}}{2}\right]$$

$$= \frac{1}{4}\left(\frac{2\pi - 3\sqrt{3}}{6}\right) \text{ or } \underline{\frac{2\pi - 3\sqrt{3}}{24}}$$

Question 13

a) i) show that:

$\sqrt{3}\cos\theta + \sin\theta = 2\cos(\theta - \pi/6)$

RHS $= 2\cos(\theta - \pi/6)$

$= 2(\cos\theta\cos\pi/6 + \sin\theta\sin\pi/6)$

$= 2\cos\theta\left(\frac{\sqrt{3}}{2}\right) + 2\sin\theta\left(\frac{1}{2}\right)$

$= \sqrt{3}\cos\theta + \sin\theta$

$=$ LHS

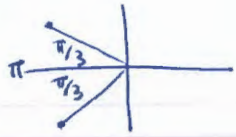
ii) Solving $\sqrt{3}\cos\theta + \sin\theta = -1$

is equivalent to solving

$2\cos(\theta - \pi/6) = -1$

$\cos(\theta - \pi/6) = -1/2$

related $\angle = \pi/3$



$\theta - \frac{\pi}{6} = 2\pi/3$ or $\theta - \pi/6 = 4\pi/3$

$\theta = \frac{2\pi}{3} + \frac{\pi}{6}$ $\qquad \theta = 4\pi/3 + \pi/6$

$\therefore \theta = \frac{5\pi}{6}$ or $\theta = 3\pi/2$

iii) From part ii), the smallest positive solution to

$\sqrt{3}\cos\theta + \sin\theta = -1$ is $\theta = 5\pi/6$

$\therefore$ The smallest positive solution to

$\sqrt{3}\cos\left(\frac{t}{3}\right) + \sin\left(\frac{t}{3}\right) = -1$ is when

$t/3 = 5\pi/6$

$t = \frac{5\pi}{6} \times 3$

$\therefore t = \frac{5\pi}{2}$ seconds

iv) Using $x = 2\cos\left(\frac{t}{3} - \pi/6\right)$

when $t = 0$: $x = 2\cos\left(\frac{0}{3} - \pi/6\right)$

$= 2\cos(-\pi/6)$

$= 2\cos\pi/6$

$= 2\frac{\sqrt{3}}{2}$

$\therefore$ initial position $= \sqrt{3}$

Now we see that the amplitude $= 2$ $\therefore$ particle oscillates between $x = -2$ and $+2$

However we need to find the initial direction of motion

i.e. Find velocity when $t = 0$

$\dot{x} = -\frac{1}{3} \times 2\sin\left(\frac{t}{3} - \pi/6\right)$

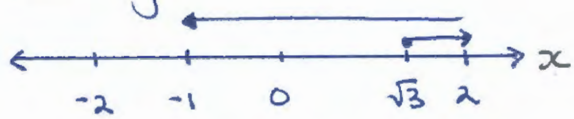
and when $t = 0$:

$\dot{x} = -\frac{2}{3}\sin(0 - \pi/6)$

$= +\frac{2}{3}\sin(\pi/6)$

> 0

$\therefore$ Particle initially moves to the right



$\therefore$ Distance travelled

$= (2 - \sqrt{3}) + 3$

$= 5 - \sqrt{3}$ metres

b) $v = \dfrac{1}{e^x}$

i.e. $\dfrac{dx}{dt} = \dfrac{1}{e^x}$

Taking reciprocals of both sides:

$$\frac{dt}{dx} = e^x$$

Taking integrals of both sides:

$$\int \frac{dt}{dx} = \int e^x \, dx$$

$$t = e^x + c$$

When $t=0$, $x=0$

$\therefore$ $0 = e^0 + c$

$0 = 1 + c$

$\therefore$ $c = -1$

So $t = e^x - 1$

$e^x = t + 1$

$\ln e^x = \ln(t+1)$

$\therefore$ $x = \ln(t+1)$

c) Prove $\displaystyle\sum_{r=1}^{n} \log\left(\frac{r+1}{r}\right) = \log(n+1)$

by Mathematical Induction

Prove true for $n=1$:

LHS $= \log\left(\frac{1+1}{1}\right)$ RHS $= \log(1+1)$

$= \log 2$ $= \log 2$

$\therefore$ since LHS = RHS, it is true for $n=1$.

Assume true for $n=k$:

i.e. $\displaystyle\sum_{r=1}^{k} \log\left(\frac{r+1}{r}\right) = \log(k+1)$

i.e. $\log\left(\frac{2}{1}\right) + \log\left(\frac{3}{2}\right) + \log\left(\frac{3}{4}\right) + \ldots + \log\left(\frac{k+1}{k}\right)$

$= \log(k+1)$ *

Prove true for $n = k+1$:

RTP: $\displaystyle\sum_{r=1}^{k+1} \log\left(\frac{r+1}{r}\right) = \log(k+2)$

LHS $= \underbrace{\log\left(\frac{2}{1}\right) + \log\left(\frac{3}{2}\right) + \ldots + \log\left(\frac{k+1}{k}\right)} + \log\left(\frac{k+2}{k+1}\right)$

$= \log(k+1) + \log\left(\frac{k+2}{k+1}\right)$ using * assumption

$= \log\left[(k+1)\left(\frac{k+2}{k+1}\right)\right]$

$= \log(k+2)$

$=$ RHS

$\therefore$ The proposition is proved true by Mathematical Induction.

Question 14

a) i) $y = \frac{1}{4}x^2 \Rightarrow 4y = x^2$

$$V = \pi \int_0^h 4y \, dy$$

$$= \pi \left[\frac{4y^2}{2} \right]_0^h$$

$$= \pi \left[2h^2 - 2(0)^2 \right]$$

$\therefore$ $V = 2\pi h^2$ (as required)

ii) $\frac{dV}{dh} = 4\pi h$

$$\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt}$$

$$\frac{dh}{dt} = \frac{1}{4\pi h} \times 2$$

$$\therefore \frac{dh}{dt} = \frac{1}{2\pi h}$$

when $h = 0.5$m,

$$\frac{dh}{dt} = \frac{1}{2\pi(0.5)}$$

$$= \frac{1}{\pi} \text{ m/min}$$

iii) Using $V = 2\pi h^2$

when $h = 3$:

$$V = 2\pi(3)^2$$

$$V = 18\pi \text{ m}^2$$

Now given the fill rate is 2m^3 per minute the time to fill the container

$$= 18\pi \div 2$$

$$= \underline{9\pi \text{ minutes}}$$

b) i) Using

$$\boxed{x = \frac{lx_1 + kx_2}{k+l}} \text{ and } \boxed{y = \frac{ly_1 + ky_2}{k+l}}$$

where $k : l = 3 : -1$

and $(x_1, y_1) = S(0, 1)$

and $(x_2, y_2) = P(2t, t^2)$

$$x = \frac{-1(0) + 3(2t)}{3 + -1} \quad y = \frac{-1(1) + 3(t^2)}{3 + -1}$$

$$x = \frac{6t}{2} = 3t \quad \text{and } y = \frac{3t^2 - 1}{2}$$

$\therefore$ M is $\left(3t, \frac{3t^2-1}{2}\right)$ as required.

ii) From the coordinates of M,

$x = 3t$ i.e. $t = \frac{x}{3}$ ... (1)

$y = \frac{3t^2 - 1}{2}$ ... (2)

sub. (1) into (2):

$$y = \frac{3\left(\frac{x}{3}\right)^2 - 1}{2}$$

$$2y = 3\left(\frac{x^2}{9}\right) - 1$$

$$2y + 1 = \frac{x^2}{3}$$

$$3(2y+1) = x^2$$

$\therefore$ $x^2 = 6y + 3$ as required.

c) i) Given $x = Vt\cos\theta$ ... ①
and $y = Vt\sin\theta - 5t^2$ ... ②

If the point (p, h) satisfies the equations, then substituting $x = p$ into ①:

$$p = Vt\cos\theta$$

i.e. $$t = \frac{p}{V\cos\theta}$$

and substituting $y = h$ into ②:

$$h = Vt\sin\theta - 5t^2$$

$$\therefore h = V\left(\frac{p}{V\cos\theta}\right)\sin\theta - 5\left(\frac{p}{V\cos\theta}\right)^2$$

$$h = p\tan\theta - \frac{5p^2}{V^2}\sec^2\theta$$

$$h = p\tan\theta - \frac{5p^2}{V^2}(1 + \tan^2\theta)$$

$$\frac{5p^2}{V^2}(1 + \tan^2\theta) = p\tan\theta - h$$

$$5p^2(1 + \tan^2\theta) = V^2(p\tan\theta - h)$$

$$\therefore V^2 = \frac{5p^2(1 + \tan^2\theta)}{p\tan\theta - h}$$

(as required)

ii) Since the point (q, h) satisfies the equations too, then $$V^2 = \frac{5q^2(1 + \tan^2\theta)}{q\tan\theta - h}$$

Equating expressions for V^2:

$$\frac{5p^2(1 + \tan^2\theta)}{p\tan\theta - h} = \frac{5q^2(1 + \tan^2\theta)}{q\tan\theta - h}$$

$$\frac{p^2}{p\tan\theta - h} = \frac{q^2}{q\tan\theta - h}$$

$$p^2(q\tan\theta - h) = q^2(p\tan\theta - h)$$

$$p^2q\tan\theta - p^2h = pq^2\tan\theta - q^2h$$

$$p^2q\tan\theta - pq^2\tan\theta = p^2h - q^2h$$

$$pq\tan\theta(p - q) = h(p - q)(p + q)$$

$$pq\tan\theta = h(p + q)$$

$$\therefore \tan\theta = \frac{h(p + q)}{pq}$$

(as required)